Singular Solutions of the Briot-Bouquet Type Partial Differential Equations (Microlocal Analysis and Related Topics)

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1 Introduction

In this talk, we will study the following tyฺpe of nonlinear singular first order partial differential equations:

$$t\partial_t u = F(t,x,u,\partial_x u) \tag{1.1}$$

where $(t,x)=(t,x_1,\ldots,x_n)\in \mathbf{C}_t\times\mathbf{C}_x^n$, $\partial_x u=(\partial_1 u,\ldots,\partial_n u)$, $\partial_t=\dfrac{\partial}{\partial t}$, $\partial_i=\dfrac{\partial}{\partial x_i}$ for $i=1,\ldots,n$, and $F(t,x,u,v)$ with $v=(v_1,\ldots,v_n)$ is a function defined in a polydisk Δ centered at the origin of $\mathbf{C}_t\times\mathbf{C}_x^n\times\mathbf{C}_u\times\mathbf{C}_v^n$. Let us denote $\Delta_0=\Delta\cap\{t=0,u=0,v=0\}$.

The assumptions are as follows:

(A1) $F(t,x,u,v)$ is holomorphic in Δ,
(A2) $F(0,x,0,0)=0$ in Δ_0,
(A3) $\dfrac{\partial F}{\partial v_i}(0,x,0,0)=0$ in Δ_0 for $i=1,\ldots,n$.

Definition 1.1 ([2], [3]) *If the equation* (1.1) *satisfies* (*A*1), (*A*2) *and* (*A*3) *we say that the equation* (1.1) *is of Briot-Bouquet type with respect to t.*

Definition 1.2 ([2], [3]) *Let us define*

$$\rho(x)=\frac{\partial F}{\partial u}(0,x,0,0),$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1.1).

Let us denote by

1. $\mathcal{R}(\mathbf{C}\backslash\{0\})$ the universal covering space of $\mathbf{C}\backslash\{0\}$,
2. $S_\theta=\{t\in\mathcal{R}(\mathbf{C}\backslash\{0\});\ |\arg t|<\theta\}$,
3. $S(\epsilon(s))=\{t\in\mathcal{R}(\mathbf{C}\backslash\{0\});\ 0<|t|<\epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on $\mathbf{R}$,

4. $D_R = \{x \in \mathbf{C}^n;\ |x_i| < R \text{ for } i = 1, \ldots, n\}$,

5. $\mathbf{C}\{x\}$ the ring of germs of holomorphic functions at the origin of $\mathbf{C}^n$.

Definition 1.3 *We define the set $\tilde{\mathcal{O}}_+$ of all functions $u(t,x)$ satisfying the following conditions;*
1. *$u(t,x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,*
2. *there is an $a > 0$ such that for any $\theta > 0$ and any compact subset K of D_R*

$$\max_{x \in K} |u(t,x)| = O(|t|^a) \quad \text{as} \quad t \to 0 \quad \text{in} \quad S_\theta.$$

We know some results on the equation (1.1) of Briot-Bouquet type with respect to t. We concern the following result. Gérard R. and Tahara H. studied in [2] the structure of holomorphic and singular solutions of the equation (1.1) and proved the following result;

Theorem 1.4 (Gérard R. and Tahara H.) *If the equation* (1.1) *is of Briot-Bouquet type and $\rho(0) \notin \mathbf{N}^* = \{1,2,3,\ldots\}$ then we have;*
(1) (*Holomorphic solutions*) *The equation* (1.1) *has a unique solution $u_0(t,x)$ holomorphic near the origin of $\mathbf{C} \times \mathbf{C}^n$ satisfying $u_0(0,x) \equiv 0$.*
(2) (*Singular solutions*) *Denote by S_+ the set of all $\tilde{\mathcal{O}}_+$-solutions of* (1.1).

$$S_+ = \begin{cases} \{u_0(t,x)\} & \text{when} \quad Re\rho(0) \leq 0, \\ \{u_0(t,x)\} \cup \{U(\varphi); 0 \neq \varphi(x) \in \mathbf{C}\{x\}\} & \text{when} \quad Re\rho(0) > 0, \end{cases}$$

where $U(\varphi)$ is an $\tilde{\mathcal{O}}_+$-solution of (1.1) *having an expansion of the following form:*

$$U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{i+2j \geq k+2, j \geq 1} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

The purpose of this paper is to determine S_+ in the case $\rho(0) \in \mathbf{N}^*$.

The main result of this paper is;

Theorem 1.5 *If the equation* (1.1) *is of Briot-Bouquet type and if $\rho(0) = N \in \mathbf{N}^*$ and $\rho(x) \not\equiv \rho(0)$, then*

$$S_+ = \{U(\varphi);\ \varphi(x) \in \mathbf{C}\{x\}\},$$

where $U(\varphi)$ is an $\tilde{\mathcal{O}}_+$-solution of (1.1) *having an expansion of the following form:*

$$U(\varphi) = u_1^0(x)t + u_0^{e_0}(x)\phi_N(t,x) + \sum_{\substack{i+|\beta| \geq 2, |\beta| < \infty \\ [\beta] \leq i+|\beta|-2}} u_i^\beta(x) t^i \Phi_N^\beta$$
$$+ w_{0,1,0}^0(x) t^{\rho(x)} + \sum_{\substack{i+j+|\beta| \geq 2, |\beta| < \infty \\ j \geq 1, [\beta] \leq i+j+|\beta|-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_N^\beta,$$

where $u_N^0(x) \equiv 0$, $w_{0,1,0}^0(x) = \varphi(x)$ *is an arbitrary holomorphic function and the other coefficients* $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ *are holomorphic functions determined by* $w_{0,1,0}^0(x)$ *and defined in a common disk, and*

$$l = (l_1, \dots, l_n) \in \mathbf{N}^n,\ |l| = l_1 + \cdots + l_n,\ \beta = (\beta_l \in \mathbf{N};\ l \in \mathbf{N}^n),$$
$$|\beta| = \sum_{|l|\geq 0} \beta_l,\ |\beta|_p = \sum_{|l|=p} \beta_l \ \mathit{for}\ p \geq 0,\ [\beta] = \sum_{|l|\geq 2} (|l|-1)\beta_l,$$
$$\Phi_N^\beta = \prod_{|l|\geq 0} \left(\frac{\partial_x^l \phi_N}{l!}\right)^{\beta_l},\ \partial_x^l = \partial_1^{l_1} \cdots \partial_n^{l_n},\ \phi_N(t,x) = \frac{t^{\rho(x)} - t^N}{\rho(x) - N}.$$

The following lemma will play an important role in the proof of Theorem 1.5. At first, we define some notations. We set for $l \in \mathbf{N}^n$, $e_l = (\beta_k;\ k \in \mathbf{N}^n)$ with $\beta_l = 1$ and $\beta_k = 0$ for $k \neq l$ and for $p \in \{1, 2, \dots, n\}$, $e(p) = (i_1, \dots, i_n)$ with $i_p = 1$ and $i_q = 0$ for $q \neq p$, and define $l^1 < l^0$ is defined by $|l^1| < |l^0|$ and $l_i^1 \leq l_i^0$ for $i = 1, \dots, n$.

Lemma 1.6 *Let* $\rho(x)$, ϕ_N *and* Φ_N^β *be as in Theorem 1.5. Then we have;*
1. $\partial_p \Phi_N^\beta = \sum_{|l|\geq 0} \beta_l (l_p + 1) \Phi_N^{\beta - e_l + e_{l+e(p)}}$ *for* $i = 1, \dots, n$,
2. $t\partial_t \phi_N = \rho(x)\phi_N + t^N$,
3. $t\partial_t \Phi_N^\beta = |\beta|\rho(x)\Phi_N^\beta + \beta_0 t^N \Phi_N^{\beta - e_0} + \sum_{|l^0|\geq 1} \sum_{l^1 < l^0} \beta_{l^0} \frac{\partial_x^{(l^0 - l^1)!}\rho(x)}{l^0 - l^1} \Phi_N^{\beta - e_{l^0} + e_{l^1}}$.

2 Construction of formal solutions in the case $\rho(0) = 1$

By [2] (Gérard-Tahara), if the equation (1.1) is of Briot-Bouquet type with respect to t, then it is enough to consider the following equation:

$$Lu = t\partial_t u - \rho(x)u = a(x)t + G_2(x)(t, u, \partial_x u) \tag{2.1}$$

where $\rho(x)$ and $a(x)$ are holomorphic functions in a neighborhood of the origin, and the function $G_2(x)(t, X_0, X_1, \dots, X_n)$ is a holomorphic function in a neighborhood of the origin in $\mathbf{C}_x^n \times \mathbf{C}_t \times \mathbf{C}_{X_0} \times \mathbf{C}_{X_1} \times \cdots \times \mathbf{C}_{X_n}$ with the following expansion:

$$G_2(x)(t, X_0, X_1, \dots, X_n) = \sum_{p+|\alpha|\geq 2} a_{p,\alpha}(x) t^p \{X_0\}^{\alpha_0} \{X_1\}^{\alpha_1} \cdots \{X_n\}^{\alpha_n}$$

and we may assume that the coefficients $\{a_{p,\alpha}(x)\}_{p+|\alpha|\geq 2}$ are holomorphic functions on D_{R_0} for a sufficiently small $R_0 > 0$. Let $0 < R < R_0$. We put $A_{p,\alpha}(R) := \max_{x \in D_R} |a_{p,\alpha}(x)|$ for $p + |\alpha| \geq 2$. Then for $0 < r < R$

$$\sum_{p+|\alpha|\geq 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t^p X_0^{\alpha_0} X_1^{\alpha_1} \times \cdots \times X_n^{\alpha_n} \tag{2.2}$$

is convergent in a neighborhood of the origin.

In this section, we assume $\rho(0) = 1$ and $\rho(x) \not\equiv 1$ and we will construct formal solutions of the equation (2.1). In generally, we set $u(t,x) = \sum_{i=1}^{N-1} u_i(x)t^i + t^{N-1}w(t,x)$, and we consider an equation for $w(t,x)$.

Proposition 2.1 *If $\rho(0) = 1$ and $\rho(x) \not\equiv 1$, the equation (2.1) has a family of formal solutions of the form:*

$$\begin{aligned} u = u_0^{e_0}(x)\phi_1 &+ \sum_{m\geq 2} \sum_{\substack{i+|\beta|=m \\ [\beta]\leq m-2}} u_i^\beta(x) t^i \Phi_1^\beta \\ &+ w_{0,1,0}^0(x)t^{\rho(x)} + \sum_{m\geq 2} \sum_{\substack{i+j+|\beta|=m \\ j\geq 1, [\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x) t^{i+j\rho(x)}\{\log t\}^k \Phi_1^\beta \end{aligned} \tag{2.3}$$

where $w_{0,1,0}^0(x)$ is an arbitrary holomorphic function and the other coefficients $u_i^\beta(x)$, $w_{i,j,k}^\beta(x)$ are holomorphic functions determined by $w_{0,1,0}^0(x)$ and defined in a common disk.

Remark 2.2 *By the relation $[\beta] \leq m-2$ in summations of the above formal solution, we have $\beta_l = 0$ for any $l \in \mathbf{N}^n$ with $|l| \geq m$.*

We define the following two sets U_m and W_m for $m \geq 1$ to prove Proposition 2.1.

Definition 2.3 *We denote by U_m the set of all functions u_m of the following forms:*

$$\begin{aligned} u_1 &= u_1^0(x)t + u_0^{e_0}(x)\phi_1, \\ u_m &= \sum_{\substack{i+|\beta|=m \\ [\beta]\leq m-2}} u_i^\beta(x)t^i\Phi_1^\beta \quad \text{for } m \geq 2, \end{aligned} \tag{2.4}$$

and denote by W_m the set of all functions w_m of the following forms:

$$\begin{aligned} w_1 &= w_{0,1,0}^0(x)t^{\rho(x)}, \\ w_m &= \sum_{\substack{i+j+|\beta|=m \\ j\geq 1, [\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)}\{\log t\}^k\Phi_1^\beta \quad \text{for } m \geq 2 \end{aligned} \tag{2.5}$$

where $u_i^\beta(x)$, $w_{i,j,k}^\beta(x) \in \mathbf{C}\{x\}$.

We can rewrite the formal solution (2.3) as follows:

$$u = \sum_{m\geq 1}(u_m + w_m) \quad \text{where } u_m \in U_m,\ w_m \in W_m.$$

Let us show important relations of u_m and w_m for $m \geq 2$. By Lemma 1.6,we have

$$
\begin{aligned}
Lw_m &= \sum_{\substack{i+j+|\beta|=m \\ j\geq 1,[\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \Big\{ \{i+(j+|\beta|-1)\rho(x)\} w^{\beta}_{i,j,k}(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta} \\
&+ k w^{\beta}_{i,j,k}(x) t^{i+j\rho(x)} \{\log t\}^{k-1} \Phi_1^{\beta} + \beta_0 w^{\beta}_{i,j,k}(x) t^{i+j\rho(x)+1} \{\log t\}^k \Phi_1^{\beta-e_0} \\
&\qquad + \sum_{|l^0|=1}^{m-1} \sum_{l^1<l^0} \beta_{l^0} \frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0-l^1)!} w^{\beta}_{i,j,k}(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta-e_{l^0}+e_{l^1}} \Big\}.
\end{aligned}
$$

We show two lemmas.

Lemma 2.4 *If $u_m \in U_m$ and $w_m \in W_m$, then $Lu_m \in U_m$ and $Lw_m \in W_m$.*

Lemma 2.5 *If $u_m \in U_m$ and $w_m \in W_m$, then the following relations hold for $i, j = 1, \ldots, n$*
1. *$a(x)U_m \subset U_m$ and $a(x)W_m \subset W_m$ for any holomorphic function $a(x)$,*
2. *tU_m, $\phi_1 U_m \subset U_{m+1}$ and $t^{\rho(x)}U_m$, tW_m, $t^{\rho(x)}W_m$, $\phi_1 W_m \subset W_{m+1}$,*
3. *$u_m \times u_n$, $\partial_i u_m \times \partial_j u_n$, $\partial_i u_m \times u_n \in U_{m+n}$,*
4. *$w_m \times w_n$, $\partial_i w_m \times \partial_j w_n$, $\partial_i w_m \times w_n$, $\in W_{m+n}$,*
5. *$u_m \times w_n$, $\partial_i u_m \times w_n$, $u_m \times \partial_j w_n$, $\partial_i u_m \times \partial_j w_n \in W_{m+n}$.*

Let us show that u_m and w_m are determined inductively on $m \geq 1$. By substituting $\sum_{m\geq 1}(u_m + w_m)$ into (2.1), we have

$$(1-\rho(x))u_1^0(x) + u_0^{e_0}(x) = a(x), \tag{2.6}$$

and for $m \geq 2$

$$Lu_m = \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_{0,h_0}} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}}, \tag{2.7}$$

$$
\begin{aligned}
Lw_m = & \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} (u_{m_{0,h_0}} + w_{m_{0,h_0}}) \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j (u_{m_{j,h_j}} + w_{m_{j,h_j}}) \\
& - \sum_{\substack{p+|\alpha|\geq 2 \\ p+|m_n|=m}} a_{p,\alpha}(x) t^p \prod_{h_0=1}^{\alpha_0} u_{m_0,h_0} \prod_{j=1}^{n} \prod_{h_j=1}^{\alpha_j} \partial_j u_{m_{j,h_j}},
\end{aligned} \tag{2.8}
$$

where $|m_n| = \sum_{i=0}^n m_i(\alpha_i)$ and $m_i(\alpha_i) = m_{i,1} + \cdots + m_{i,\alpha_i}$ for $i = 0, 1, \ldots, n$.

We take any holomorphic function $\varphi(x) \in \mathbf{C}\{x\}$ and put $w^0_{0,1,0}(x) = \varphi(x)$, and by (2.6), we put $u_1^0(x) \equiv 0$ and $u_0^{e_0}(x) = a(x)$.

For $m \geq 2$, let us show that u_m and w_m are determined by induction. By Lemma 2.5, the right side of (2.7) belongs to U_m and the right side of (2.8) belongs to W_m. Further by $m_{j,h_j} \geq 1$, we have $m_{j,h_j} < m$ for $h_j = 1, \ldots, \alpha_j$ and $j = 0, \ldots, n$. Then for $m \geq 2$, we compare with the coefficients of $t^i \Phi_1^\beta$ and $t^{i+j\rho(x)}\{\log t\}^k \Phi_1^\beta$ respectively for (2.7) and (2.8), then put

$$\begin{aligned}
&\{i + (|\beta| - 1)\rho(x)\}u_i^\beta(x) \\
&+ (\beta_0 + 1)u_{i-1}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0 \leq l^1 < l^0} (\beta_{l^0} + 1)\frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0 - l^1)!}u_i^{\beta+e_{l^0}-e_{l^1}}(x) \\
&= f_i^\beta(\{a_{p,\alpha}\}_{2\leq p+|\alpha|\leq m},\ \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'|<m})
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\{i + (j + |\beta| - 1)\rho(x)\}w_{i,j,k}^\beta(x) + (k+1)w_{i,j,k+1}^\beta(x) \\
&+ (\beta_0 + 1)w_{i-1,j,k}^{\beta+e_0}(x) + \sum_{|l^0|=1}^{m-1} \sum_{0 \leq l^1 < l^0} (\beta_{l^0} + 1)\frac{\partial_x^{l^0-l^1}\rho(x)}{(l^0 - l^1)!}w_{i,j,k}^{\beta+e_{l^0}-e_{l^1}}(x) \\
&= g_{i,j,k}^\beta(\{a_{p,\alpha}\}_{2\leq p+|\alpha|\leq m},\ \{u_{i'}^{\beta'}(x)\}_{i'+|\beta'|<m},\ \{w_{i',j',k'}^{\beta'}(x)\}_{i'+j'+|\beta'|<m}).
\end{aligned} \tag{2.10}$$

Hence we obtain Proposition 2.1. Q.E.D.

3 Convergence of the formal solutions in the case $\rho(0) = 1$

In this section, we show that the formal solution (2.3) converges in $\widetilde{\mathcal{O}}_+$.

Proposition 3.1 *Let γ satisfy $0 < \gamma < 1$ and let λ be sufficiently large. Then for any sufficiently small $r > 0$ we have the following result;*

For any $\theta > 0$ there is an $\epsilon > 0$ such that the formal solution (2.3) converges in the following region:

$$\begin{aligned}
\{(t,x) \in \mathbf{C}_t \times \mathbf{C}_x^n;\ &|\eta(t,\lambda)t| < \epsilon,\ |\eta(t,\lambda)^2 t^{\rho(x)}| < \epsilon, \\
&|\eta(t,\lambda)t^\gamma| < \epsilon,\ t \in S_\theta \text{ and } x \in D_r\},
\end{aligned}$$

where $\eta(t,\lambda) = \max\{|(\log t)/\lambda|,\ 1\}$.

In this section, we put $w_{i,0,0}^\beta(x) = u_i^\beta(x)$ and $w_{i,0,k}^\beta(x) \equiv 0$ for $k \geq 1$ in the formal solution (2.3). Then the formal solution (2.3) is as follows:

$$\begin{aligned}
u = &\, w_{0,0,0}^{e_0}(x)\phi_1 + w_{0,1,0}^0(x)t^{\rho(x)} \\
&+ \sum_{m\geq 2} \sum_{\substack{i+j+|\beta|=m \\ [\beta]\leq m-2}} \sum_{\substack{k\leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} w_{i,j,k}^\beta(x)t^{i+j\rho(x)}\{\log t\}^k \Phi_1^\beta.
\end{aligned} \tag{3.1}$$

Let us define the following set V_m for (3.1).

Definition 3.2 *We denote by V_m the set of all the functions v_m of the following forms:*

$$v_m = u_m + w_m \quad for \quad u_m \in U_m \quad and \quad w_m \in W_m. \tag{3.2}$$

We define the following estimate for the function v_m.

Definition 3.3 *For the function* (3.2), *we define*

$$\begin{aligned} ||v_1||_{r,c,\lambda} &= ||v_1||_{r,c} := \frac{||w^{e_0}_{0,0,0}||_r}{c} + ||w^0_{0,1,0}||_r, \\ ||v_m||_{r,c,\lambda} &:= \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{||w^{\beta}_{i,j,k}||_r \lambda^k}{c^{<\beta>}} \quad for \quad m \geq 2 \end{aligned} \tag{3.3}$$

for $c > 0$ and $\lambda > 0$, where

$$||w^{\beta}_{i,j,k}||_r = \max_{x\in D_r} |w^{\beta}_{i,j,k}(x)| \ and \ <\beta>= \sum_{|l|\ge 0} (|l|+1)\beta_l.$$

We will make use of

Lemma 3.4 *For a holomorphic function $f(x)$ on D_{R_0}, we have*

$$||\partial^{\alpha}_x f||_R \le \frac{\alpha!}{(R_0 - R)^{|\alpha|}} ||f||_{R_0} \quad for \quad 0 < R < R_0.$$

Proof. By Cauchy's integral formula, we have the desired result. Q.E.D

Lemma 3.5 *If a holomorphic function $f(x)$ on D_R satisfies*

$$||f||_r \le \frac{C}{(R-r)^p} \quad for \quad 0 < r < R$$

then we have

$$||\partial_i f||_r \le \frac{Ce(p+1)}{(R-r)^{p+1}} \quad for \quad 0 < r < R, \quad i = 1, \dots, n.$$

For the proof, see Hörmander ([5]lemma 5.1.3)

Let us show the following estimate for the function Lv_m.

Lemma 3.6 *Let $0 < R < R_0$. Then there exists a positive constant σ such that for $m \geq 2$, if $v_m \in V_m$ we have*

$$||Lv_m||_{r,c,\lambda} \ge \frac{\sigma}{2} m ||v_m||_{r,c,\lambda} \quad for \quad 0 < r \le R$$

for sufficiently small $c > 0$ and sufficiently large $\lambda > 0$.

Let us estimate the function $\partial_i v_m$.

Definition 3.7 *For the function $v_m \in V_m$ we define*

$$D_p v_m := \sum_{\substack{i+j+|\beta|=m \\ [\beta]\le m-2}} \sum_{\substack{k\le i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \partial_p w^{\beta}_{i,j,k}(x) t^{i+j\rho(x)} \{\log t\}^k \Phi_1^{\beta}$$

for $p = 1, \ldots, n$.

Lemma 3.8 *If $v_m \in V_m$, then for $i = 1, \ldots, n$, we have*

$$||\partial_i v_m||_{r,c,\lambda} \le ||D_i v_m||_{r,c,\lambda} + c_0 \lambda m ||v_m||_{r,c,\lambda} + \frac{3m-2}{c} ||v_m||_{r,c,\lambda} \quad \textit{for} \quad 0 < r \le R. \tag{3.4}$$

Therefore by the relations (2.7), (2.8) and Lemma 3.8, we have the following lemma.

Lemma 3.9 *If $u = \sum_{m\ge1} v_m$ is a formal solution of the equation* (2.1) *constructed in Section 2, we have the following inequality for v_m $(m \ge 2)$:*

$$\begin{aligned} ||Lv_m||_{r,c,\lambda} \le & \sum_{\substack{p+|\alpha|\ge 2 \\ p+|m_n|=m}} ||a_{p,\alpha}||_r \prod_{h_0=1}^{\alpha_0} ||v_{m_{0,h_0}}||_{r,c,\lambda} \\ & \times \prod_{i=1}^{n} \prod_{h_i=1}^{\alpha_i} \{||D_i v_{m_{i,h_i}}||_{r,c,\lambda} + c_0 \lambda m_{i,h_i} ||v_{m_{i,h_i}}||_{r,c,\lambda} + \frac{3m_{i,h_i}-2}{c} ||v_{m_{i,h_i}}||_{r,c,\lambda}\}. \end{aligned}$$

Let us define a majorant equation to show that the formal solution (3.1) converges.
We take A_1 so that

$$\frac{||w^{eo}_{0,0,0}||_R}{c} + ||w^{0}_{0,1,0}||_R \le A_1,$$
$$\frac{||\partial_i w^{eo}_{0,0,0}||_R}{c} + ||\partial_i w^{0}_{0,1,0}||_R \le A_1$$

for $i = 1, \ldots, n$.

Then we consider the following equation:

$$\begin{aligned} \frac{\sigma}{2} Y = & \frac{\sigma}{2} A_1 t_1 \\ & + \frac{1}{R-r} \sum_{p+|\alpha|\ge 2} \frac{A_{p,\alpha}(R)}{(R-r)^{p+|\alpha|-2}} t_1{}^p Y^{\alpha_0} \prod_{i=1}^{n} \left(eY + c_0 \lambda Y + \frac{3}{c} Y\right)^{\alpha_i}. \end{aligned} \tag{3.5}$$

The equation (3.5) has a unique holomorphic solution $Y = Y(t_1)$ with $Y(0) = 0$ at $(Y, t_1) = (0, 0)$ by implicit function theorem. By an easy calculation, the solution $Y = Y(t_1)$ has the following form:

$$Y = \sum_{m \geq 1} Y_m {t_1}^m \text{ with } Y_m = \frac{C_m}{(R-r)^{m-1}}$$

where $Y_1 = C_1 = A_1$ and $C_m \geq 0$ for $m \geq 1$.
Then we have;

Lemma 3.10 *For $m \geq 1$, we have*

$$m||v_m||_{r,c,\lambda} \leq Y_m \quad \text{for} \quad 0 < r < R. \tag{3.6}$$

Let us show that the formal solution (3.1) converges by using (3.6) in Lemma 3.10. We rewrite v_m as follows:

$$v_m = \sum_{\substack{i+j+|\beta|=m \\ [\beta] \leq m-2}} \sum_{\substack{k \leq i+|\beta|_0+|\beta|_1 \\ +2(j-1)}} \frac{w^{\beta}_{i,j,k}(x)\lambda^k}{c^{<\beta>}} t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right)^k \Psi^{\beta}_1,$$

where

$$\Psi^{\beta}_1 = \prod_{|l| \geq 0} \left(c^{|l|+1} \frac{\partial^l_x \phi_1}{l!}\right)^{\beta_l}. \tag{3.7}$$

Firstly let us estimate (3.7). For $||\phi_1||_R$, we have the following lemma.

Lemma 3.11 *For any γ with $0 < \gamma < 1$, there is an $R > 0$ such that*

$$||\phi_1||_R = O\left(|t|^{\gamma}\right) \text{ as } t \to 0 \text{ in } S_{\theta}$$

holds for any $\theta > 0$.

By Lemma 3.11, there exists a positive constant c_1 such that

$$||\phi_1||_R \leq c_1 |t|^{\gamma} \quad \text{in} \quad S_{\theta}. \tag{3.8}$$

By Lemma 3.4 and (3.8), we have

$$||\Psi^{\beta}_1||_r \leq \prod_{|l| \geq 0} \left(c^{|l|+1} \frac{c_1}{(R-r)^{|l|}} |t|^{\gamma}\right)^{\beta_l} = \left(\frac{c}{R-r}\right)^{<\beta>} (c_1(R-r)|t|^{\gamma})^{|\beta|} \tag{3.9}$$

for $0 < r < R < R_0$ in S_{θ}.

Let us estimate $t^{i+j\rho(x)} \left(\frac{\log t}{\lambda}\right)^k \Psi^{\beta}_1$.

We put $\eta(t, \lambda) = \max\left\{\left|\frac{\log t}{\lambda}\right|,\ 1\right\}$, $c_2 = \max\left\{\frac{c}{R-r}, 1\right\}$ and $c_3 = c_1(R-r)$. Since we have $[\beta] \le m-2 < m = i+j+|\beta|$, $<\beta> \le 2|\beta| + [\beta] \le i+j+3|\beta|$ and $k \le i + |\beta|_0 + |\beta|_1 + 2(j-1) \le i + |\beta| + 2j$, we obtain

$$\left\|t^{i+j\rho(x)}\left(\frac{\log t}{\lambda}\right)^k \Psi_1^\beta\right\|_r \le \{|c_2\eta(t,\lambda)t|\}^i \left\{||c_2\eta(t,\lambda)^2 t^{\rho(x)}||_r\right\}^j \left\{|(c_2)^3 c_3\eta(t,\lambda)t^\gamma|\right\}^{|\beta|}$$

in S_θ. For any sufficiently small $\epsilon > 0$, there exists a sufficiently small $\delta > 0$ such that for any $t \in S_\theta$ with $0 < |t| < \delta$ we have

$$|c_2\eta(t,\lambda)t| < \epsilon, \ ||c_2\eta(t,\lambda)^2 t^{\rho(x)}||_r < \epsilon, \ |(c_2)^3 c_3\eta(t,\lambda)t^\gamma| < \epsilon.$$

Then by Lemma 3.10, we have

$$||u||_r \le \sum_{m\ge 1} Y_m \epsilon^m \tag{3.10}$$

for sufficiently small $|t|$ in S_θ. Hence the formal solution (3.1) converges for $x \in D_r$ and sufficiently small $|t|$ in S_θ. Q.E.D.

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